Hamiltonian dynamics of vortex and magnetic lines in hydrodynamic type systems

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(Received 4 May 1999)

Vortex line and magnetic line representations are introduced for a description of flows in ideal hydrodynamics and magnetohydrodynamics (MHD), respectively. For incompressible fluids, it is shown with the help of this transformation that the equations of motion for vorticity $\mathbf{\Omega}$ and magnetic field follow from a variational principle. By means of this representation, it is possible to integrate the hydrodynamic type system with the Hamiltonian $\mathcal{H}=\int|\mathbf{\Omega}|d\mathbf{r}$ and some other systems. It is also demonstrated that these representations allow one to remove from the noncanonical Poisson brackets, defined in the space of divergence-free vector fields, the degeneracy connected with the vorticity frozenness for the Euler equation and with magnetic field frozenness for ideal MHD. For MHD, a new Weber-type transformation is found. It is shown how this transformation can be obtained from the two-fluid model when electrons and ions can be considered as two independent fluids. The Weber-type transformation for ideal MHD gives the whole Lagrangian vector invariant. When this invariant is absent, this transformation coincides with the Clebsch representation analog introduced by V.E. Zakharov and E. A. Kuznetsov [Dokl. Ajad. Nauk **194**, 1288 (1970) [Sov. Phys. Dokl. **15**, 913 (1971)]].

PACS number(s): 52.30.−q, 52.35.Ra, 52.55.Fa

I. INTRODUCTION

There are a large number of works devoted to the Hamiltonian description of ideal hydrodynamics (see, e.g., the review [2] and the references therein). This question was first studied by Clebsch (a citation can be found in Ref. [3]), who introduced for nonpotential flows of incompressible fluids a pair of variables λ and μ (which later were called Clebsch variables). Fluid dynamics in these variables is such that vortex lines are represented by the intersection of surfaces $\lambda=\text{const}$ and $\mu=\text{const}$. These quantities, being canonically conjugated variables, remain constant by fluid advection. However, these variables, as known (see, e.g., [4]), describe only a partial type of flows. If λ and μ are single-valued functions of coordinates, then the linking degree of vortex lines characterized by the Hopf invariant [5] is equal to zero. For arbitrary flows, the Hamiltonian formulation of the equation for incompressible ideal hydrodynamics was given by Arnold [6,7]. The Euler equations for the vorticity $\mathbf{\Omega}=\text{curl}\,\mathbf{v}$,

$$\frac{\partial\mathbf{\Omega}}{\partial t}=\text{curl}[\mathbf{v}\times\mathbf{\Omega}],\qquad \text{div}\,\mathbf{v}=\mathbf{0}, \tag{1}$$

are written in the Hamiltonian form

$$\frac{\partial\mathbf{\Omega}}{\partial t}=\{\mathbf{\Omega},\mathcal{H}\}, \tag{2}$$

by means of the noncanonical Poisson brackets [4]

$$\{F,G\}=\int\left(\mathbf{\Omega}\left[\text{curl}\frac{\delta F}{\delta\mathbf{\Omega}}\times\text{curl}\frac{\delta G}{\delta\mathbf{\Omega}}\right]\right)d\mathbf{r}, \tag{3}$$

where the Hamiltonian

$$\mathcal{H}_h=-\frac{1}{2}\int\mathbf{\Omega}\Delta^{-1}\mathbf{\Omega}\,d\mathbf{r}, \tag{4}$$

coincides with the total fluid energy.

In spite of the fact that the bracket (3) allows us to describe flows with arbitrary topology, the main drawback is its degeneracy. For this reason it is impossible to formulate the variational principle on the whole space $\mathcal{S}$ of divergence-free vector fields.

The cause of the degeneracy, namely, the presence of Casimirs annulling the Poisson bracket, is connected with the existence of a special symmetry formed by the whole group—the relabeling group of Lagrangian markers. This fact was first understood completely by Salmon in 1982 [8], although Eckart in 1938 and then in 1960 [9] and later Newcomb [10] understood the role of this symmetry. All known theorems about the vorticity conservation (Ertel's, Cauchy's, and Kelvin's theorems, the frozenness of vorticity and conservation of the topological Hopf invariant) are a sequence of this symmetry. The main one of these theorems is the frozenness of vortex lines into the fluid. This is related to the local Lagrangian invariant—the Cauchy invariant. The physical meaning of this invariant consists in that any fluid particle always remains on its own vortex line.

A similar situation takes place also for ideal magnetohydrodynamics (MHD) for barotropic fluids:

$$\rho_t+\nabla\cdot(\rho\mathbf{v})=\mathbf{0}, \tag{5}$$

$$\mathbf{v}_t+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla w(\rho)+\frac{1}{4\pi\rho}[\text{curl}\,\mathbf{h}\times\mathbf{h}], \tag{6}$$

$$\mathbf{h}_t=\text{curl}[\mathbf{v}\times\mathbf{h}]. \tag{7}$$

Here ρ is a plasma density, $w(\rho)$ is plasma enthalpy, $\mathbf{v}$ is velocity, and $\mathbf{h}$ is magnetic field. As is well known (see, e.g.,

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[11–15]), the MHD equations possess one important feature—frozenness of magnetic field into plasma which is destroyed only due to dissipation (by finite conductivity). For ideal MHD, a combination of the continuity equation (5) and the induction equation (7) gives the analog of the Cauchy invariant for MHD.

The MHD equations of motion (5)–(7) can be also represented in the Hamiltonian form,

$$\rho_t=\{\rho,\mathcal{H}\},\quad \mathbf{h}_t=\{\mathbf{h},\mathcal{H}\},\quad \mathbf{v}_t=\{\mathbf{v},\mathcal{H}\},$$

by means of the noncanonical Poisson brackets [16],

$$\{F,G\}=\int\left(\frac{\mathrm{curl}\,\mathbf{v}}{\rho}\left[\frac{\delta F}{\delta\mathbf{v}}\times\frac{\delta G}{\delta\mathbf{v}}\right]\right)d\mathbf{r}+\int\left\{\frac{\mathbf{h}}{\rho}\left(\left[\mathrm{curl}\frac{\delta F}{\delta\mathbf{h}}\times\frac{\delta G}{\delta\mathbf{v}}\right]-\left[\mathrm{curl}\frac{\delta G}{\delta\mathbf{h}}\times\frac{\delta F}{\delta\mathbf{v}}\right]\right)\right\}d\mathbf{r}+\int\left[\frac{\delta G}{\delta\rho}\nabla\left(\frac{\delta F}{\delta\mathbf{v}}\right)-\frac{\delta F}{\delta\rho}\nabla\left(\frac{\delta G}{\delta\mathbf{v}}\right)\right]d\mathbf{r}. \tag{8}$$

This bracket is also degenerate. For instance, the integral $\int(\mathbf{v},\mathbf{h})d\mathbf{r}$, which characterizes the linking number of vortex and magnetic lines, is one of the Casimirs for this bracket.

The analog of the Clebsch representation in MHD serves a change of variables suggested in 1970 by Zakharov and one of the authors of this paper (E.K.) [1]:

$$\mathbf{v}=\nabla\phi+\frac{[\mathbf{h}\times\mathrm{curl}\,\mathbf{S}]}{\rho}. \tag{9}$$

The new variables (ϕ,ρ) and $(\mathbf{h},\mathbf{S})$ represent two pairs of canonically conjugate quantities with the Hamiltonian coinciding with the total energy,

$$\mathcal{H}=\int\left(\rho\frac{\mathbf{v}^2}{2}+\rho\tilde{\varepsilon}(\rho)+\frac{\mathbf{h}^2}{8\pi}\right)d\mathbf{r},$$

where $\tilde{\varepsilon}(\rho)$ is a specific internal energy.

In the present paper we suggest an approach of resolving the degeneracy of the noncanonical Poisson brackets for incompressible fluids by introducing new variables, namely, Lagrangian markers labeling vortex lines for ideal hydrodynamics or magnetic lines in the MHD case.

The basis of this approach is an integral representation for the corresponding frozen-in solenoidal field, namely, the vorticity for the Euler equation and the magnetic field for MHD. We introduce new objects, i.e., the vortex lines or magnetic lines, and we obtain the equations of motion for them. This description is a mixed Lagrangian-Eulerian description, when each vortex (or magnetic) line is enumerated by a Lagrangian marker, but motion along the line is described in terms of the Eulerian variables. Such representation fixes all topological invariants of solenoidal field. It removes the degeneracy from the Poisson brackets connected with the conservation of all topological properties, retaining the gauge invariance of the equations of motion with respect to reparametrization of each line. It is important that the equations for line motion, as the equations for curve deformation, are transverse to the line tangent.

It is interesting that the line representation also solves another problem—the equations of line motion follow from the variational principle, being Hamiltonian.

This approach allows us also to consider the limit of narrow vortex (or magnetic) lines. For two-dimensional flows in hydrodynamics, this ‘‘new’’ description corresponds to a well-known fact, namely, the canonical conjugation of x and y coordinates of vortices (see, e.g., [3]).

The Hamiltonian structure introduced makes it possible to integrate the infinite set of three-dimensional Euler equations (2) with local Hamiltonians depending on a curvature κ and a torsion χ of vortex lines: $\mathcal{H}_1=\int|\mathbf{\Omega}|d\mathbf{r},\mathcal{H}_2=\int|\mathbf{\Omega}|\chi d\mathbf{r},\mathcal{H}_3=\int|\mathbf{\Omega}|(\kappa^2/2)d\mathbf{r}$, and so on. In terms of the vortex lines, the given Hamiltonians are decomposed into a set of Hamiltonians of noninteracting vortex lines. The dynamics of each vortex line is, in turn, described by an equation which can be reduced by the Hasimoto transformation [17] to an integrable equation from the hierarchy of the one-dimensional nonlinear Schrödinger equation.

For ideal MHD, a new representation—an analog of the Weber transformation—is found. This representation contains the whole vector Lagrangian invariant. In the case of ideal hydrodynamics, this invariant provides conservation of the Cauchy invariant and, as a sequence, all known conservation laws for vorticity (for details, see the review [2]). It is important that all these conservation laws can be expressed in terms of observable variables. Unlike the Euler equation, these vector Lagrangian invariants for the MHD case cannot be expressed in terms of density, velocity, and magnetic field. It is necessary to tell that the analog of the Weber transformation for MHD includes the change of variables (9) as a partial case. The presence of these Lagrangian invariants in the transform provides topologically nontrivial MHD flows.

The Weber transform and its analog for MHD play a key role in constructing the vortex line (or magnetic line) representation. This representation is based on the property of frozenness. Therefore, by means of such a transform the noncanonical Poisson brackets become nondegenerated in these variables and, as a result, a variational principle may be formulated. Another peculiarity of this representation is its locality, establishing the correspondence between vortex (or magnetic) line and vorticity (or magnetic field). This is a specific mapping, mixed Lagrangian-Eulerian, for which the Jacobian of the mapping cannot be equal to unity for incompressible fluids as it is for pure Lagrangian description.

II. GENERAL REMARKS

We start our consideration from some well known facts, namely, the Lagrangian description of ideal hydrodynamics.

In the Eulerian description for barotropic fluids [pressure depends on density only: $p=p(\rho)$], we have coupled equations—continuity equation for density ρ and the Euler equation for velocity $\mathbf{v}$:

$$\rho_t+\mathrm{div}(\rho\mathbf{v})=\mathbf{0}, \tag{10}$$

$$\mathbf{v}_t+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla w(\rho),\quad dw(\rho)=dp/\rho. \tag{11}$$

In the Lagrangian description, each fluid particle has its own label. This is a three-dimensional vector $\mathbf{a}$, so that the particle position at time t is given by the function

$$\mathbf{x}=\mathbf{x}(\mathbf{a},t). \tag{12}$$

Usually the initial position of a particle serves as the Lagrangian marker: $\mathbf{a}=\mathbf{x}(\mathbf{a},0)$.

In the Lagrangian description, the Euler equation (11) is nothing more than the Newton equation:

$$\ddot{\mathbf{x}}=-\nabla w.$$

In this equation the second derivative with respect to time t is taken for fixed $\mathbf{a}$, but the right-hand side of the equation is a function of t and $\mathbf{x}$. Excluding from the latter the x dependence, the Euler equation takes the form

$$\ddot{x}_i \frac{\partial x_i}{\partial a_k}=-\frac{\partial w(\rho)}{\partial a_k}, \tag{13}$$

where now all quantities are functions of t and $\mathbf{a}$.

In the Lagrangian description, the continuity equation (10) is easily integrated and the density is given through the Jacobian of the mapping (12) $J=\det(\partial x_i/\partial a_k)$,

$$\rho=\rho_0(\mathbf{a})/J. \tag{14}$$

Now let us introduce a new vector,

$$u_k=\frac{\partial x_i}{\partial a_k}v_i, \tag{15}$$

which has the meaning of velocity in a new curvilinear system of coordinates. Alternatively, one may say that this formula defines the transformation law for velocity components. It is worth noting that Eq. (15) gives the transformation for the velocity $\mathbf{v}$ as a *covector*.

A straightforward calculation gives that the vector $\mathbf{u}$ satisfies the equation

$$\frac{du_k}{dt}=\frac{\partial}{\partial a_k}\left(\frac{\mathbf{v}^2}{2}-w\right). \tag{16}$$

In this equation the right-hand side represents the gradient relative to $\mathbf{a}$ and therefore the ''transverse'' part of the vector $\mathbf{u}$ will be conserved in time. And this gives the Cauchy invariant [18],

$$\frac{d}{dt}\mathrm{curl}_a\mathbf{u}=\mathbf{0}, \quad \text{or} \quad \mathrm{curl}_a\mathbf{u}=\mathbf{I}. \tag{17}$$

If Lagrangian markers $\mathbf{a}$ are initial positions of fluid particles, then the Cauchy invariant coincides with the initial vorticity: $\mathbf{I}=\mathbf{\Omega}_0(\mathbf{a})$. This invariant is expressed through the instantaneous value of $\mathbf{\Omega}(\mathbf{x},t)$ by the relation

$$\mathbf{\Omega}_0(\mathbf{a})=J\cdot[\mathbf{\Omega}(\mathbf{x},t)\nabla]\mathbf{a}(\mathbf{x},t), \tag{18}$$

where $\mathbf{a}=\mathbf{a}(\mathbf{x},t)$ is the inverse mapping to Eq. (12). Following from Eq. (18), the relation for $\mathbf{B}=\mathbf{\Omega}/\rho$,

$$B_{0i}(a)=\frac{\partial a_i}{\partial x_k}B_k(x,t),$$

shows that, unlike velocity, $\mathbf{B}$ transforms as a vector.

By integrating Eq. (16) over time t, we arrive at the so-called Weber transformation (see, e.g., [3]):

$$\mathbf{u}(\mathbf{a},t)=\mathbf{u}_0(\mathbf{a})+\nabla_a\mathbf{\Phi}, \tag{19}$$

where the potential $\mathbf{\Phi}$ obeys the Bernoulli equation:

$$\frac{d\mathbf{\Phi}}{dt}=\frac{\mathbf{v}^2}{2}-w(\rho) \tag{20}$$

with the initial condition $\mathbf{\Phi}|_{t=0}=\mathbf{0}$. For such a choice of $\mathbf{\Phi}$ a new function $\mathbf{u}_0(\mathbf{a})$ is connected with the ''transverse'' part of $\mathbf{u}$ by the evident relation

$$\mathrm{curl}_a\mathbf{u}_0(\mathbf{a})=\mathbf{I}.$$

The Cauchy invariant $\mathbf{I}$ characterizes how the vorticity is frozen into the fluid. It can be obtained in the standard way by considering two equations—the equation for the quantity $\mathbf{B}=\mathbf{\Omega}/\rho$,

$$\frac{d\mathbf{B}}{dt}=(\mathbf{B}\cdot\nabla)\mathbf{v}, \tag{21}$$

and the equation for the vector $\delta\mathbf{x}=\mathbf{x}(\mathbf{a}+\delta\mathbf{a})-\mathbf{x}(\mathbf{a})$ between two adjacent fluid particles,

$$\frac{d\,\delta\mathbf{x}}{dt}=(\delta\mathbf{x}\cdot\nabla)\mathbf{v}. \tag{22}$$

The comparison of these two equations shows that if initially the vectors $\delta\mathbf{x}$ are parallel to the vector $\mathbf{B}$, then they will be parallel to each other for all time. This is merely the statement that vorticity is frozen into the fluid. Each fluid particle always remains at its own vortex line. The combination of Eqs. (21) and (22) leads to the Cauchy invariant. To establish this fact, it is enough to write down the equation for the Jacoby matrix $J_{ij}=\partial x_i/\partial a_j$, which directly follows from Eq. (22):

$$\frac{d}{dt}\frac{\partial a_i}{\partial x_k}=-\frac{\partial a_i}{\partial x_j}\frac{\partial v_j}{\partial x_k}.$$

This equation, in combination with Eq. (21), gives conservation of the Cauchy invariant (17).

If now one comes back to the velocity field $\mathbf{v}$, then by use of Eqs. (15) and (19) it is possible to obtain that

$$\mathbf{v}=u_{0k}\nabla a_k+\nabla\mathbf{\Phi}, \tag{23}$$

where the gradient is taken with respect to $\mathbf{x}$. Here the equation for the potential $\mathbf{\Phi}$ has the standard form of the Bernoulli equation:

$$\mathbf{\Phi}_t+(\mathbf{v}\cdot\nabla)\mathbf{\Phi}-\frac{\mathbf{v}^2}{2}+w(\rho)=0.$$

It is interesting to note that relations (17), as equations for determination of $\mathbf{x}(\mathbf{a},t)$, unlike Eqs. (16), are of first order with respect to the time derivative. This fact is also reflected in the expression for the velocity (23), which can be considered as a result of the partial integration of the equations of motion (16). Of course, the velocity field given by Eq. (23) contains two unknown functions: one is the vector $\mathbf{a}(\mathbf{x},t)$ and another is the potential $\mathbf{\Phi}$. For incompressible fluids the latter is determined from the condition div $\mathbf{v}=\mathbf{0}$. In this case the Bernoulli equation determines the pressure.

Another important point concerning the Cauchi invariant is that it follows from the invariance of the variational principle—the action is unchanged under the relabeling transformation (for details, see the reviews [8,2]). Passing from the Lagrangian to the Hamiltonian in this description we have no problems with the Poisson bracket. It is given in the standard way and does not contain any degeneracy against the noncanonical Poisson brackets (3) and (8). One of the main purposes of this paper is to construct a new description of the Euler equation (as well as ideal MHD) which, on the one hand, would allow us to retain the Eulerian description, as maximally as possible, but, on the other hand, would include from the very beginning the frozenness connected with the relabeling symmetry.

As for MHD, this system has a common feature with the Euler equation: it also possesses the frozenness property. The equation for $\mathbf{h}/\rho$ coincides with Eq. (21) and therefore the dynamics of magnetic lines is very similar to that for vortex lines of the Euler equation. However, this analogy cannot be continued so far because the equation of motion for velocity differs from the Euler equation by the presence of ponderomotive force. This difference remains also for the incompressible case.

III. VORTEX LINE REPRESENTATION

Consider the Hamiltonian dynamics of the divergence-free vector field $\mathbf{\Omega}(\mathbf{r},t)$, given by the Poisson bracket (3) with some Hamiltonian $\mathcal{H}$:

$$\frac{\partial\mathbf{\Omega}}{\partial t}=\mathrm{curl}\left[\mathrm{curl}\frac{\delta\mathcal{H}}{\delta\mathbf{\Omega}}\times\mathbf{\Omega}\right]. \qquad (24)$$

Notice that substitution of Eq. (4) into Eq. (24) yields the Euler equation (1).

As we have said, the bracket (3) is degenerate, as a result of which it is impossible to formulate the variational principle on the entire space $\mathcal{S}$ of solenoidal vector fields. It is known [2] that Casimirs f, annulling Poisson brackets, distinguish invariant manifolds $\mathcal{M}_f$ (symplectic leaves) in $\mathcal{S}$ on each of which it is possible to introduce standard Hamiltonian mechanics and accordingly to write down a variational principle. We shall show that solution of this problem for Eq. (24) is possible on the basis of the property of frozenness of the field $\mathbf{\Omega}(\mathbf{r},t)$, which allows us to resolve all the constraints, stipulated by the Casimirs, and gives the necessary formulation of the variational principle.

To each Hamiltonian $\mathcal{H}$—functional of $\mathbf{\Omega}(\mathbf{r},t)$—we associate the velocity field

$$\tilde{\mathbf{v}}(\mathbf{r},t)=\mathrm{curl}\frac{\delta\mathcal{H}}{\delta\mathbf{\Omega}}. \qquad (25)$$

However, one should note that the generalized velocity $\mathbf{v}(\mathbf{r},t)$ can be defined up to the addition of the vector parallel to $\mathbf{\Omega}$:

$$\mathbf{v}=\tilde{\mathbf{v}}+\alpha\mathbf{\Omega}. \qquad (26)$$

The substitution $\tilde{\mathbf{v}}\to\mathbf{v}$ in no way changes Eq. (24) for $\mathbf{\Omega}$. Hence, it becomes clear that instead of the transformation $\mathbf{x}=\mathbf{x}(\mathbf{a},t)$ of the initial positions of fluid particles $\mathbf{x}(\mathbf{a},0)=\mathbf{a}$ by the velocity field $\tilde{\mathbf{v}}(\mathbf{r},t)$ through the solution of the equation

$$\dot{\mathbf{x}}=\tilde{\mathbf{v}}(\mathbf{x},t), \qquad (27)$$

some other transformation can be used. The possible transformations are defined by the generalized velocity $\mathbf{v}$ (26) and correspond to the various choices of the α function. Therefore, using a full Lagrangian description for the systems (24) becomes ineffective.

Now we introduce the following general expression for $\mathbf{\Omega}(\mathbf{r})$, which is gauge invariant and fixes all topological properties of the system that are determined by the initial field $\mathbf{\Omega}_0(\mathbf{a})$ [19]:

$$\mathbf{\Omega}(\mathbf{r},t)=\int\delta(\mathbf{r}-\mathbf{R}(\mathbf{a},t))[\mathbf{\Omega}_0(\mathbf{a})\cdot\nabla_{\mathbf{a}}]\mathbf{R}(\mathbf{a},t)d\mathbf{a}. \qquad (28)$$

Here now

$$\mathbf{r}=\mathbf{R}(\mathbf{a},t) \qquad (29)$$

does not satisfy Eq. (27) any more and, consequently, the mapping Jacobian $J=\det\|\partial\mathbf{R}/\partial\mathbf{a}\|$ is not assumed to equal 1, as it was for the full Lagrangian description of incompressible fluids.

It is easy to check that from the condition $[\nabla_{\mathbf{a}}\mathbf{\Omega}_0(\mathbf{a})]=\mathbf{0}$ it follows that divergence of Eq. (28) is identically equal to zero.

The gauge transformation

$$\mathbf{R}(\mathbf{a})\to\mathbf{R}(\tilde{\mathbf{a}}_{\Omega_0}(\mathbf{a})) \qquad (30)$$

leaves this integral unchanged if $\tilde{\mathbf{a}}_{\Omega_0}$ arises from $\mathbf{a}$ by means of arbitrary nonuniform translations along the field line of $\mathbf{\Omega}_0(\mathbf{a})$. Therefore, the invariant manifold $\mathcal{M}_{\Omega_0}$ of the space $\mathcal{S}$, on which the variational principle holds, is obtained from the space $\mathcal{R}$: $\mathbf{a}\to\mathbf{R}$ of arbitrary continuous one-to-one three-dimensional mappings identifying $\mathcal{R}$ elements that are obtained from one another with the help of the gauge transformation (30) with a fixed solenoidal field $\mathbf{\Omega}_0(\mathbf{a})$.

It is important also that $\mathbf{\Omega}_0(\mathbf{a})$ can be expressed explicitly in terms of the instantaneous value of the vorticity and the mapping $\mathbf{a}=\mathbf{a}(\mathbf{r},t)$, inverse to Eq. (29). By integrating over the variables $\mathbf{a}$ in the relation (28),

$$\mathbf{\Omega}(\mathbf{R})=\frac{[\mathbf{\Omega}_0(\mathbf{a})\cdot\nabla_{\mathbf{a}}]\mathbf{R}(\mathbf{a})}{\det\|\partial\mathbf{R}/\partial\mathbf{a}\|}, \qquad (31)$$

where $\mathbf{\Omega}_0(\mathbf{a})$ can be represented in the form

$$\mathbf{\Omega}_0(\mathbf{a})=\det\|\partial\mathbf{R}/\partial\mathbf{a}\|[\mathbf{\Omega}(\mathbf{r})\cdot\nabla]\mathbf{a}.$$

This formula is merely the Cauchy invariant (17). We note that according to Eq. (31) the vector

$$\mathbf{b}(\mathbf{a},t)=[\mathbf{\Omega}_0(\mathbf{a})\cdot\nabla_{\mathbf{a}}]\mathbf{R}(\mathbf{a},t) \tag{32}$$

is tangent to $\mathbf{\Omega}(\mathbf{R})$. It is natural to introduce parameter s as an arc length of the initial vortex lines $\mathbf{\Omega}_0(\mathbf{a})$ so that

$$\mathbf{b}=\Omega_0(\nu)\frac{\partial\mathbf{R}}{\partial s}.$$

In this expression Ω_0 depends on the transverse parameter ν labeling each vortex line. In accordance with this, the representation (28) can be written in the form

$$\mathbf{\Omega}(\mathbf{r},t)=\int\Omega_0(\nu)d^2\nu\int\delta(\mathbf{r}-\mathbf{R}(s,\nu,t))\frac{\partial\mathbf{R}}{\partial s}ds, \tag{33}$$

from which the meaning of the new variables becomes clearer: To each vortex line with index ν there is associated the curve $\mathbf{r}=\mathbf{R}(s,\nu,t)$, and the integral (33) itself is a sum over vortex lines. We notice that the parametrization by introduction of s and ν is local. Globally, therefore, the representation (33) can be used only for distributions with closed vortex lines.

To get the equation of motion for $\mathbf{R}(\mathbf{a},t)$, the representation (28) must be substituted in the Euler equation (24). Using the formula

$$\mathbf{\Omega}_t(\mathbf{r},t)=\mathrm{curl}_{\mathbf{r}}\int\delta(\mathbf{r}-\mathbf{R}(\mathbf{a},t))[\mathbf{R}_t(\mathbf{a},t)\times\mathbf{b}(\mathbf{a},t)]d\mathbf{a},$$

which follows from Eq. (28), one can obtain

$$\mathrm{curl}_{\mathbf{r}}\left(\frac{\mathbf{b}(\mathbf{a},t)\times[\mathbf{R}_t(\mathbf{a},t)-\mathbf{v}(\mathbf{R},t)]}{\det\|\partial\mathbf{R}/\partial\mathbf{a}\|}\right)=0.$$

This equation can be solved by setting the expression under the curl operator equal identically to zero:

$$[\mathbf{b}\times\mathbf{R}_t(\mathbf{a},t)]=[\mathbf{b}\times\mathbf{v}(\mathbf{R},t)] \tag{34}$$

or, in terms of coordinates ν and s,

$$[\mathbf{R}_s\times\mathbf{R}_t(\nu,s,t)]=[\mathbf{R}_s\times\mathbf{v}(\mathbf{R},t)].$$

With this choice there remains the freedom both to change the parameter s and relabel the transverse coordinates ν. Notice that, as it follows from Eqs. (III) and (31), a motion of a point on the manifold $\mathcal{M}_{\Omega_0}$ is determined only by the component of the generalized velocity transverse to $\mathbf{\Omega}(\mathbf{r})$.

The obtained equation Eq. (III) is the equation of motion for vortex lines. In accordance with Eq. (III), the evolution of each vector $\mathbf{R}$ is principally transverse to the vortex line. The longitudinal component of velocity has no effect on the line dynamics.

The description of vortex lines with the help of Eqs. (33) and (III) is a mixed Lagrangian-Eulerian one: The parameter ν has a clear Lagrangian origin whereas the coordinate s remains Eulerian.

IV. VARIATIONAL PRINCIPLE

The key observation for formulation of the variational principle is that the following general equality holds for functionals that depend only on $\mathbf{\Omega}$:

$$\left[\mathbf{b}\times\mathrm{curl}\left(\frac{\delta F}{\delta\mathbf{\Omega}(\mathbf{R})}\right)\right]=\frac{\delta F}{\delta\mathbf{R}(\mathbf{a})}\Big|_{\mathbf{\Omega}_0}. \tag{35}$$

For this reason, the right-hand side of (III) equals the variational derivative $\delta\mathcal{H}/\delta\mathbf{R}$:

$$[\mathbf{\Omega}_0(\mathbf{a})\cdot\nabla_{\mathbf{a}}]\mathbf{R}(\mathbf{a})\times\mathbf{R}_t(\mathbf{a})=\frac{\delta\mathcal{H}\{\mathbf{\Omega}\{\mathbf{R}\}\}}{\delta\mathbf{R}(\mathbf{a})}\Big|_{\mathbf{\Omega}_0}. \tag{36}$$

It is not difficult to check now that, as described by Eq. (36), the dynamics of vortex lines is equivalent to the requirement of an extremum of the action ($\delta S=0$) with the Lagrangian [19],

$$\mathcal{L}=\frac{1}{3}\int\{[\mathbf{R}_t(\mathbf{a})\times\mathbf{R}(\mathbf{a})]\cdot[\mathbf{\Omega}_0(\mathbf{a})\cdot\nabla_{\mathbf{a}}]\mathbf{R}(\mathbf{a})\}d\mathbf{a}-\mathcal{H}\{\mathbf{\Omega}\{\mathbf{R}\}\}. \tag{37}$$

Thus, we have introduced a variational principle for Hamiltonian dynamics of divergence-free vector field topologically equivalent to $\mathbf{\Omega}_0(\mathbf{a})$. The Lagrangian (37) has a remaining symmetry connected with relabeling of Lagrangian markers of vortex lines. This symmetry leads to conservation of volumes inside all closed vortex surfaces. This property explains why the Jacobian of the mapping $\mathbf{r}=\mathbf{R}(\mathbf{a},t)$ cannot be equal identically to unity.

Let us discuss some properties of the equations of motion (36), which are associated with the excess parametrization of elements of $\mathcal{M}_{\Omega_0}$ by objects from $\mathcal{R}$. We want to pay attention to the fact that from Eq. (35) there follows the property that the vectors $\mathbf{b}$ and $\delta F/\delta\mathbf{R}(\mathbf{a})$ are orthogonal for all functionals defined on $\mathcal{M}_{\Omega_0}$. In other words, the variational derivative of the gauge-invariant functionals should be understood [specifically, in Eq. (35)] as

$$\hat{P}\frac{\delta F}{\delta\mathbf{R}(\mathbf{a})},$$

where $\hat{P}_{ij}=\delta_{ij}-\tau_i\tau_j$ is a projector and $\boldsymbol{\tau}=\mathbf{b}/|\mathbf{b}|$ a unit vector tangent to the vortex line. Using this property as well as the transformation formula (35), it is possible, by a direct calculation of the bracket (3), to obtain the Poisson bracket (between two gauge-invariant functionals) expressed in terms of vortex lines:

$$\{F,G\}=\int\frac{d\mathbf{a}}{|\mathbf{b}|^2}\left(\mathbf{b}\left[\hat{P}\frac{\delta F}{\delta\mathbf{R}(\mathbf{a})}\times\hat{P}\frac{\delta G}{\delta\mathbf{R}(\mathbf{a})}\right]\right). \tag{38}$$

The new bracket (38) does not contain variational derivatives with respect to $\mathbf{\Omega}_0(\mathbf{a})$. Therefore, with respect to the initial bracket, the Cauchy invariant $\mathbf{\Omega}_0(\mathbf{a})$ is a Casimir fixing the invariant manifolds $\mathcal{M}_{\Omega_0}$ on which it is possible to introduce the variational principle (37).

In the case of the hydrodynamics of a superfluid liquid, a Lagrangian of the form (37) was apparently first used by

Rasetti and Regge [20] to derive an equation of motion, identical to Eq. (III), but for a separate vortex filament. Later, on the base of the results [20], Volovik and Dotsenko, Jr. [21] obtained the Poisson bracket between the coordinates of the vortices and the velocity components for a continuous distribution of vortices. The expression for these brackets can be extracted without difficulty from the general form for the Poisson brackets (38). However, the noncanonical Poisson brackets obtained in [20,21] must be used with care. Their direct application gives for the equation of motion of the coordinate of a vortex filament an answer that is not gauge-invariant. For a general time-dependent variation, additional terms describing flow along a vortex appear in the equation of motion. For this reason, the dynamics of curves (including vortex lines) is in principle ''transverse'' with respect to the curve itself.

Sometimes it is possible to parametrize lines by one of the Cartesian coordinates (for instance, the z coordinate),

$$\mathbf{R}(\nu,z,t)=(X(\nu,z,t),Y(\nu,z,t),z).$$

For this case, functions X and Y are canonically conjugated quantities. The Lagrangian (37) in the case of ideal incompressible hydrodynamics takes the form

$$\mathcal{L}=\int d^2\nu dz\dot{X}Y - \frac{1}{8\pi}\int\int \frac{(1+X_1'X_2'+Y_1'Y_2')dz_1dz_2d^2\nu_1d^2\nu_2}{\sqrt{(z_1-z_2)^2+(X_1-X_2)^2+(Y_1-Y_2)^2}}. \tag{39}$$

Here the double integral is the Hamiltonian (4), $X_1' = \partial X(\nu_1,z_1,t)/\partial z_1$, and so on.

V. INTEGRABLE HYDRODYNAMICS

Now we present an example of the equations of the hydrodynamic type (24), for which transition to the representation of vortex lines permits us to establish the fact of their integrability [19].

First let us consider the Hamiltonian

$$\mathcal{H}\{\mathbf{\Omega}(\mathbf{r})\}=\int|\mathbf{\Omega}|d\mathbf{r} \tag{40}$$

and the corresponding equation of frozenness (24) with the generalized velocity

$$\mathbf{v}=\mathrm{curl}(\mathbf{\Omega}/\Omega).$$

We assume that vortex lines are closed and apply the representation (33). Then due to Eq. (31), the Hamiltonian in terms of vortex lines is decomposed as a sum of Hamiltonians of vortex lines:

$$\mathcal{H}\{\mathbf{R}\}=\int|\Omega_0(\nu)|d^2\nu\int\left|\frac{\partial\mathbf{R}}{\partial s}\right|ds. \tag{41}$$

This integral over s is the total length of the vortex line with index ν. According to Eq. (36), with respect to these variables the equation of motion for the vector $\mathbf{R}(\nu,s)$ is local; it does not contain terms describing interaction with other vortices:

$$\eta[\mathbf{R}_s\times\mathbf{R}_t(\nu,s,t)]=-\boldsymbol{\tau}_s\equiv[\boldsymbol{\tau}\times(\boldsymbol{\tau}\times\boldsymbol{\tau}_s)]. \tag{42}$$

Here $\eta=\mathrm{sgn}(\Omega_0)$, $\boldsymbol{\tau}=\mathbf{R}_s/|\mathbf{R}_s|$ is the unit vector tangent to the vortex line.

This equation is invariant against changes $s\to\tilde{s}(s,t)$. Therefore, Eq. (42) can be solved for $\mathbf{R}_t$ up to a shift along the vortex line—the transformation does not change the vorticity $\mathbf{\Omega}$. This means that to find $\mathbf{\Omega}$ it is enough to have one solution of the equation

$$\eta|\mathbf{R}_s|\mathbf{R}_t=[\boldsymbol{\tau}\times\boldsymbol{\tau}_s]+\beta\mathbf{R}_s, \tag{43}$$

which follows from Eq. (42) for some value of β. This leads to an equation for $\boldsymbol{\tau}$ as a function of filament length $l(dl=|\mathbf{R}_s|ds)$ and time t (by choosing a new value $\beta=0$), which reduces to the integrable one-dimensional Landau-Lifshits equation for a Heisenberg ferromagnet:

$$\eta\frac{\partial\boldsymbol{\tau}}{\partial t}=\left[\boldsymbol{\tau}\times\frac{\partial^2\boldsymbol{\tau}}{\partial l^2}\right].$$

This equation is gauge-equivalent to the one-dimensional (1D) nonlinear Schrödinger equation [22],

$$i\psi_t+\psi_{ll}+\frac{1}{2}|\psi|^2\psi=0,$$

and, for instance, can be reduced to the NLSE by means of the Hasimoto transformation [17,23]:

$$\psi(l,t)=\kappa(l,t)\exp\left(i\int^l\chi(\tilde{l},t)d\tilde{l}\right),$$

where $\kappa(l,t)$ is a curvature and $\chi(l,t)$ the line torsion.

The system with the Hamiltonian (40) has direct relation to hydrodynamics. As known (see the paper [17], and references therein), the local approximation for the thin vortex filament (under assumption of smallness of the filament width to the characteristic longitudinal scale) leads to the Hamiltonian (41) but only for one separate line. Respectively, Eq. (24) with the Hamiltonian (40) can be used for a description of motion of several vortex filaments, whose thickness is small compared with a distance between them. In this case the (nonlinear) dynamics of each filament is independent of the behavior of its neighbors. In the framework of this model, the appearance of singularities (intersection of vortices) is of an inertial character very similar to the wave breaking in gas dynamics. Of course, this approximation does not work if the distances between filaments are comparable with filament thickness.

It should be noted also that for the given approximation, the Hamiltonian of a vortex filament is proportional to its length. From its conservation, it follows that this model is inadequate for modeling the behavior of vortex filaments in turbulent flows where usually the process of vortex filament stretching takes place. It is desirable to have a better model free from this drawback. A new model must necessarily describe nonlocal effects.

In addition, we would like to say that the list of Eq. (24) which can be integrated with the help of representation (33) is not exhausted by Eq. (40). So, the system with the Hamiltonian

$$\mathcal{H}_2\{\mathbf{\Omega}(\mathbf{r})\}=\int|\mathbf{\Omega}|\chi d\mathbf{r} \tag{44}$$

is gauge equivalent to the modified KdV equation

$$\psi_t+\psi_{lll}+\frac{3}{2}|\psi|^2\psi_l=0,$$

the second one after NLSE in the hierarchy generated by the Zakharov-Shabat operator. Really the infinite set of Hamiltonians $\mathcal{H}_n\{\mathbf{\Omega}\}$ exists, so that each $\mathcal{H}_n$ corresponds to the integrable equation of order $n+2$ from this hierarchy:

$$\mathcal{H}_3=\int|\mathbf{\Omega}|\frac{\kappa^2}{2}d\mathbf{r},\quad \mathcal{H}_4=\int|\mathbf{\Omega}|\frac{\chi\kappa^2}{2}d\mathbf{r},$$

$$\mathcal{H}_5=\int|\mathbf{\Omega}|\left(\frac{\kappa'^2+\chi^2\kappa^2}{2}-\frac{\kappa^4}{8}\right)d\mathbf{r},\quad \ldots .$$

As against the previous model (40), some physical application of $\mathcal{H}_n$ with $n>1$ has not yet been found.

VI. LAGRANGIAN (MATERIAL) DESCRIPTION OF MHD

Consider now how the relabeling symmetry works in ideal MHD. First, rewrite the equations of motion (5)–(7) in the Lagrangian representation by introducing markers $\mathbf{a}$ for fluid particles

$$\mathbf{x}=\mathbf{x}(\mathbf{a},t),\quad \mathbf{v}(\mathbf{x},t)=\dot{\mathbf{x}}(\mathbf{a},t).$$

In this case the continuity equation (5) and the equation for the magnetic field (7) can be integrated. The density and the magnetic field are expressed in terms of the Jacoby matrix by means of Eq. (14) and by the equation

$$B_i(x,t)=\frac{\partial x_i}{\partial a_k}B_{0k}(a), \tag{45}$$

where $\mathbf{B}=\mathbf{h}/\rho$. In the latter transformation the Jacoby matrix serves the evolution operator for vector $\mathbf{B}$. The vector $\mathbf{B}$, in turn, transforms as a vector.

In terms of Lagrangian variables, the equation of motion (6) is written as follows:

$$\frac{\partial x_i}{\partial a_k}\ddot{x}_i=-\frac{\partial w(\rho)}{\partial a_k}+\frac{J}{4\pi\rho_0(\mathbf{a})}[\mathrm{curl}\,\mathbf{h}\times\mathbf{h}]_i\frac{\partial x_i}{\partial a_k}. \tag{46}$$

With the help of relation (45) and Eq. (16), the vector $\mathbf{u}$ given by Eq. (15) will satisfy the equation

$$\frac{d\mathbf{u}}{dt}=\mathbf{\nabla}\left(\frac{\mathbf{v}^2}{2}-\mathbf{w}\right)-\frac{1}{4\pi}[\mathbf{B}_0(\mathbf{a})\times\mathrm{curl}_a\mathbf{H}]. \tag{47}$$

Here the vector $\mathbf{B}_0(\mathbf{a})=\mathbf{h}_0(\mathbf{a})/\rho_0(\mathbf{a})$ is a Lagrangian invariant and $\mathbf{H}$ represents the co-adjoint transformation of the magnetic field, analogous to Eq. (15):

$$H_i(a,t)=\frac{\partial x_m}{\partial a_i}h_m(x,t).$$

Now by analogy with Eqs. (16) and (19), integration of Eq. (47) over time leads to the Weber-type transformation,

$$\mathbf{u}(\mathbf{a},t)=\mathbf{u}_0(\mathbf{a})+\mathbf{\nabla}_a\mathbf{\Phi}+[\mathbf{B}_0(\mathbf{a})\times\mathrm{curl}_a\tilde{\mathbf{S}}]. \tag{48}$$

Here $\mathbf{u}_0(\mathbf{a})$ is a new Lagrangian invariant which can be chosen as purely transverse, namely, with $\mathrm{div}_a\mathbf{u}_0=\mathbf{0}$. This new Lagrangian invariant cannot be expressed through the observed physical quantities such as magnetic field, velocity, and density. In spite of this fact, as it will be shown in the next section, the vector Lagrangian invariant $\mathbf{u}_0(\mathbf{a})$ has a clear physical meaning. As for the new variables $\mathbf{\Phi}$ and $\tilde{\mathbf{S}}$, they obey the equations

$$\frac{d\mathbf{\Phi}}{dt}=\frac{\mathbf{v}^2}{2}-w,$$

$$\frac{d\tilde{\mathbf{S}}}{dt}=-\frac{\mathbf{H}}{4\pi}+\mathbf{\nabla}_a\psi.$$

The transformation (48) for velocity $\mathbf{v}(\mathbf{x},t)$ takes the form

$$\mathbf{v}=u_{0k}(\mathbf{a})\mathbf{\nabla}a_k+\mathbf{\nabla}\Phi+\left[\frac{\mathbf{h}}{\rho}\times\mathrm{curl}\,\mathbf{S}\right], \tag{49}$$

where $\mathbf{S}$ is the vector $\tilde{\mathbf{S}}$ transformed by means of the rule (15),

$$S_i(x,t)=\frac{\partial a_k}{\partial x_i}\tilde{S}_k(a,t).$$

In the Eulerian description,$\mathbf{\Phi}$ satisfies the Bernoulli equation

$$\frac{\partial\mathbf{\Phi}}{\partial t}+(\mathbf{v}\cdot\mathbf{\nabla})\mathbf{\Phi}-\frac{\mathbf{v}^2}{2}+w=0 \tag{50}$$

and the equation of motion for $\mathbf{S}$ is of the form

$$\frac{\partial\mathbf{S}}{\partial t}+\frac{\mathbf{h}}{4\pi}-[\mathbf{v}\times\mathrm{curl}\,\mathbf{S}]+\mathbf{\nabla}\psi_1=\mathbf{0}. \tag{51}$$

For $\mathbf{u}_0=\mathbf{0}$, the transformation (49) was introduced for ideal MHD in 1970 [1]. In this case the magnetic field $\mathbf{h}$ and vector $\mathbf{S}$ as well as $\mathbf{\Phi}$ and ρ are two pairs of canonically conjugate variables. It is interesting to note that in the canonical case the equations of motion for $\mathbf{S}$ and $\mathbf{\Phi}$ obtained in [1] coincide with Eqs. (50) and (51). However, the canonical parametrization describes only some types of flows. In particular, it does not describe topologically nontrivial flows for which the linking number between magnetic and vortex lines is not equal to zero. This topological characteristic is given by the integral $\int(\mathbf{v},\mathbf{h})d\mathbf{x}$. Only when $\mathbf{u}_0\neq\mathbf{0}$ does this integral take nonzero values.

VII. FROZEN-IN MHD FIELDS

To clarify the meaning of the new Lagrangian invariant $\mathbf{u}_0(\mathbf{a})$, we recall that the MHD equations (5)–(7) can be obtained from a two-fluid system where electrons and ions are considered as two separate fluids interacting with each other by means of a self-consistent electromagnetic field. The MHD equations follow from two-fluid equations in the low-frequency limit when characteristic frequencies are less than the ion gyrofrequency. The latter assumes (i) neglect of electron inertia, (ii) smallness of electric field with respect to magnetic field, and (iii) charge quasineutrality. We write down at first some intermediate system often called MHD with dispersion [24],

$$\mathrm{curl}\;\;\mathrm{curl}\,\mathbf{A}=\frac{4\pi e}{c}(n_1\mathbf{v}_1-n_2\mathbf{v}_2), \tag{52}$$

$$(\partial_t+\mathbf{v}_1\boldsymbol{\nabla})m\mathbf{v}_1=\frac{e}{c}(-\mathbf{A}_t+[\mathbf{v}_1\times\mathrm{curl}\,\mathbf{A}])-\boldsymbol{\nabla}\frac{\partial\varepsilon}{\partial n_1}, \tag{53}$$

$$0=-\frac{e}{c}(-\mathbf{A}_t+[\mathbf{v}_2\times\mathrm{curl}\,\mathbf{A}])-\boldsymbol{\nabla}\frac{\partial\varepsilon}{\partial n_2}. \tag{54}$$

In these equations, $\mathbf{A}$ is the vector potential so that the magnetic field $\mathbf{h}=\mathrm{curl}\,\mathbf{A}$ and electric field $\mathbf{E}=-(1/c)\mathbf{A}_t$. This system is closed by two continuity equations for ion density n_1 and electron density n_2:

$$n_{1,t}+\boldsymbol{\nabla}(n_1\mathbf{v}_1)=0,\quad n_{2,t}+\boldsymbol{\nabla}(n_2\mathbf{v}_2)=\mathbf{0}.$$

In this system $\mathbf{v}_{1,2}$ are velocities of ion and electron fluids, respectively. The first equation of this system is a Maxwell equation for a magnetic field in a static limit. The second equation is the equation of motion for the ions. The next one is the equation of motion for electrons in which we neglect the electron inertia. By means of the latter equation, one can obtain the equation of frozenness of a magnetic field into the electron fluid (this is another Maxwell equation),

$$\mathbf{h}_t=\mathrm{curl}[\mathbf{v}_2\times\mathbf{h}].$$

Applying the operator div to Eq. (52) gives, taking account of the continuity equations, the quasineutrality condition: $n_1=n_2=n$. Next, by eliminating n_2 and $\mathbf{v}_2$ we have finally the equations of MHD with dispersion in its standard form [24]:

$$(\partial_t+\mathbf{v}\cdot\boldsymbol{\nabla})m\mathbf{v}=-\boldsymbol{\nabla}w(n)+\frac{1}{4\pi n}[\mathrm{curl}\,\mathbf{h}\times\mathbf{h}],$$

$$n_t+\boldsymbol{\nabla}\cdot(n\mathbf{v})=0, \tag{55}$$

$$\mathbf{h}_t=\mathrm{curl}\left[\left(\mathbf{v}-\frac{c}{4\pi en}\mathrm{curl}\,\mathbf{h}\right)\times\mathbf{h}\right],$$

where $\mathbf{v}_1=\mathbf{v}$, and $\varepsilon(n,n)$ is the internal energy density so that $w(n)=\partial/\partial n[\varepsilon(n,n)]$ is the enthalpy for an ion-electron pair. Classical MHD follows from this system in the limit when the last term $c/(4\pi en)\mathrm{curl}\,\mathbf{h}$ in Eq. (55) can be neglected with respect to $\mathbf{v}$. At the same time, the vector potential $\mathbf{A}$ must be larger than the characteristic values of $(mc/e)\mathbf{v}$ in order to provide the same order of magnitude for inertia and magnetic terms in Eq. (53). Both requirements are satisfied if $\epsilon=c/(\omega_{pi}L)\ll 1$, where L is a characteristic scale of magnetic field variation and $\omega_{pi}=\sqrt{4\pi ne^2/m}$ is the ion plasma frequency.

Unlike the MHD equations (5)–(7), the given system has two frozen-in fields. These are the field $\mathbf{\Omega}_2=-(e/mc)\mathbf{h}$ frozen into the electron fluid and the field

$$\mathbf{\Omega}_1=\mathrm{curl}\left(\mathbf{v}+\frac{e}{mc}\mathbf{A}\right)=\mathbf{\Omega}-\mathbf{\Omega}_2$$

frozen into the ion component:

$$\mathbf{\Omega}_{1t}=\mathrm{curl}[\mathbf{v}\times\mathbf{\Omega}_1],$$
$$\mathbf{\Omega}_{2t}=\mathrm{curl}[\mathbf{v}_2\times\mathbf{\Omega}_2], \tag{56}$$

where

$$\mathbf{v}_2=\mathbf{v}-\frac{c}{4\pi en}\mathrm{curl}\,\mathbf{h}.$$

Hence for both fields one can construct two Cauchy invariants by the same rule (17) as for ideal hydrodynamics:

$$\mathbf{\Omega}_{10}(\mathbf{a})=J_1\cdot[\mathbf{\Omega}_1(\mathbf{x},t)\cdot\boldsymbol{\nabla}]\mathbf{a}(\mathbf{x},t), \tag{57}$$

where $\mathbf{a}(\mathbf{x},t)$ is the inverse mapping to $\mathbf{x}=\mathbf{x}_1(\mathbf{a},t)$, which is the solution of the equation $\dot{\mathbf{x}}_1=\mathbf{v}(\mathbf{x}_1,t)$,

$$\mathbf{\Omega}_{20}(\mathbf{a}_2)=J_2\cdot[\mathbf{\Omega}_2(\mathbf{x},t)\cdot\boldsymbol{\nabla}]\mathbf{a}_2(\mathbf{x},t) \tag{58}$$

with $\mathbf{a}_2(\mathbf{x},t)$ inverse to the mapping $\mathbf{x}=\mathbf{x}_2(\mathbf{a}_2,t)$ and $\dot{\mathbf{x}}_2=\mathbf{v}_2(\mathbf{x}_2,t)$.

In order to get the corresponding Weber transformation for MHD as a limit of the system, it is necessary to introduce two momenta for ion and electron fluids:

$$\mathbf{p}_1=m\mathbf{v}+\frac{e}{c}\mathbf{A},$$

$$\mathbf{p}_2=-\frac{e}{c}\mathbf{A}.$$

As $\epsilon\to 0$, in these expressions the terms containing the vector potential are much greater than the sum of $\mathbf{p}_1$ and $\mathbf{p}_2$. For each momentum in the Lagrangian representation one can get equations, analogous to Eqs. (13) and (16),

$$\frac{\partial x_{1k}}{\partial a_i}\dot{p}_{1k}=-p_{1k}\frac{\partial v_k}{\partial a_i}+\frac{\partial}{\partial a_i}\left[-\frac{\partial\varepsilon}{\partial n_1}+\frac{e}{c}(\mathbf{v}\cdot\mathbf{A})+m\frac{v^2}{2}\right],$$

$$\frac{\partial x_{2k}}{\partial a_{2i}}\dot{p}_{2k}=-p_{2k}\frac{\partial v_{2k}}{\partial a_{2i}}+\frac{\partial}{\partial a_{2i}}\left[-\frac{\partial\varepsilon}{\partial n_2}-\frac{e}{c}(\mathbf{v}_2\cdot\mathbf{A})\right].$$

By introducing the vector $\tilde{\mathbf{p}}$ for each type of fluid, by the same rule as Eq. (15),

$$\tilde{p}_i=\frac{\partial x_k}{\partial a_i}p_k,$$

after integration over time of the equations of motion for $\tilde{\mathbf{p}}$ one can arrive at two Weber transformations for each momentum,

$$\mathbf{p}_1 = \tilde{p}_{1i}(\mathbf{a}) \nabla a_i + \nabla \Phi_1, \tag{59}$$

$$\mathbf{p}_2 = \tilde{p}_{2i}(\mathbf{a}_2) \nabla a_{2i} + \nabla \Phi_2. \tag{60}$$

In the limit $\epsilon \to \mathbf{0}$ the markers $\mathbf{a}$ and $\mathbf{a}_2$ can be set approximately equal. This means that their difference will be small:

$$\mathbf{a}_2 - \mathbf{a} = \mathbf{d} \sim \epsilon.$$

Besides, due to charge quasineutrality, Jacobians with respect to $\mathbf{a}$ and $\mathbf{a}_2$ must be equal to each other [here we set $n_{10}(\mathbf{a}) = n_{20}(\mathbf{a}_2) = 1$ without loss of generality]:

$$\det \| \partial \mathbf{a} / \partial \mathbf{x} \| = \det \| \partial \mathbf{a}_2 / \partial \mathbf{x} \|.$$

As a result, the infinitesimal vector $\mathbf{d}(\mathbf{a},t)$ relative to the argument $\mathbf{a}$ must be divergence-free: $\partial d_i / \partial a_i = 0$.

Then, summing Eqs. (59) and (60) and considering the limit $\epsilon \to 0$, we obtain the Weber-type transformation coinciding with Eq. (48):

$$\mathbf{u}(\mathbf{a},t) = \mathbf{u}_0(\mathbf{a}) + \nabla_a \Phi + [\mathbf{B}_0(\mathbf{a}) \times \mathrm{curl}_a \tilde{\mathbf{S}}], \tag{61}$$

where vectors $\mathbf{u}_0(\mathbf{a})$ and $\tilde{\mathbf{S}}$ are expressed through the Lagrangian invariants $\tilde{\mathbf{p}}_1(\mathbf{a})$ and $\tilde{\mathbf{p}}_2(\mathbf{a})$ and displacement $\mathbf{d}$ between electron and ion by means of the relations [25]

$$\mathbf{u}(\mathbf{a},t) = \frac{1}{m} [\tilde{\mathbf{p}}_1(\mathbf{a}) + \tilde{\mathbf{p}}_2(\mathbf{a})],$$

$$\mathbf{d} = -\frac{mc}{e} \mathrm{curl}_a \tilde{\mathbf{S}}.$$

It is important that in Eq. (61) all terms are of the same order of magnitude (zero order relative to ϵ). Taking the curl of vectors $\tilde{\mathbf{p}}_1(\mathbf{a})$ and $\tilde{\mathbf{p}}_2(\mathbf{a}_2)$ yields the corresponding Cauchy invariants (57) and (58).

VIII. RELABELING SYMMETRY IN MHD

Now let us show how the existence of new Lagrangian invariants corresponds to the relabeling symmetry.

Consider the MHD Lagrangian [2],

$$\mathcal{L}_* = \int \left(\rho \frac{\mathbf{v}^2}{2} - \rho \tilde{\varepsilon}(\rho) - \frac{\mathbf{h}^2}{8\pi} \right) d\mathbf{r},$$

where we neglect the contribution from the electric field in comparison with that from the magnetic field. Here $\tilde{\varepsilon}(\rho)$ is the specific internal energy.

In terms of the mapping $\mathbf{x}(\mathbf{a},t)$, the Lagrangian $\mathcal{L}_*$ is rewritten as follows [26]:

$$\mathcal{L}_* = \int \frac{\dot{\mathbf{x}}^2}{2} d\mathbf{a} - \int \tilde{\varepsilon}(J_{\mathbf{x}}^{-1}(\mathbf{a})) d\mathbf{a} - \frac{1}{8\pi} \int \left(\frac{[\mathbf{h}_0(\mathbf{a}) \cdot \nabla_{\mathbf{a}}]\mathbf{x}}{J_{\mathbf{x}}(\mathbf{a})} \right)^2 J_{\mathbf{x}}(\mathbf{a}) d\mathbf{a}. \tag{62}$$

Here the density and the magnetic field are expressed by means of the relations

$$\rho = 1/J_{\mathbf{x}}, \quad \mathbf{h} = [\mathbf{h}_0(\mathbf{a}) \cdot \nabla_a]\mathbf{x}/J_{\mathbf{x}}$$

and

$$J_{\mathbf{x}}(\mathbf{a},t) = \det \| \partial \mathbf{x} / \partial \mathbf{a} \|$$

is the Jacobian of the mapping $\mathbf{x} = \mathbf{x}(\mathbf{a},t)$ and the initial density is set equal to 1. Notice that the variation of the action given by the Lagrangian (62) with respect to $\mathbf{x}(\mathbf{a})$ gives the equation of motion (46) [or the equivalent equation for vector $\mathbf{u}$ (47)].

Due to the presence of the magnetic field in the Lagrangian (62), the group of relabeling symmetry, in comparison with ideal hydrodynamics, is restricted. Although the first two terms in Eq. (62) are invariant with respect to all incompressible changes $\mathbf{a} \to \mathbf{a}(\mathbf{c})$ with $J|_c = 1$, the invariance of the last term, however, restricts the possible deformations to the class satisfying the condition

$$[\mathbf{h}_0(\mathbf{a}) \cdot \nabla_{\mathbf{a}}]\mathbf{c} = \mathbf{h}_0(\mathbf{c}).$$

For infinitesimal transformations

$$\mathbf{a} \to \mathbf{a} + \tau \mathbf{g}(\mathbf{a}),$$

where τ is a (small) group parameter, the vector $\mathbf{g}$ must thus satisfy two conditions:

$$\mathrm{div}_a \mathbf{g} = 0, \quad \mathrm{curl}_a [\mathbf{g} \times \mathbf{h}_0] = \mathbf{0}. \tag{63}$$

The first condition is the same as for ideal hydrodynamics, the second provides magnetic-field frozenness.

The conservation laws generated by this symmetry, in accordance with Noethers theorem, can be obtained by the standard scheme from the Lagrangian (62). They are written through the infinitesimal deformation $\mathbf{g}(\mathbf{a})$ as an integral over $\mathbf{a}$:

$$I = \int (\mathbf{u}, \mathbf{g}(\mathbf{a})) d\mathbf{a}, \tag{64}$$

where the vector $\mathbf{u}$ is given by Eq. (15). Setting $\mathbf{g} = \mathbf{h}_0$ from this (infinite) family of integrals, one gets the simplest one,

$$I_{\mathrm{ch}} = \int (\mathbf{v}, \mathbf{h}) d\mathbf{r},$$

which represents a cross-helicity characterizing the degree of mutual linking of vortex and magnetic lines.

The conservation laws (64) are compatible with the Weber-type transformation. Really, substituting Eq. (48) into Eq. (64) and using Eq. (63), one obtains the relation

$$\int (\mathbf{u}_0(\mathbf{a}), \mathbf{g}(\mathbf{a})) d\mathbf{a}.$$

Hence conservation of Eq. (64) also follows. Note that if one did not suppose $\mathbf{u}_0$ to be independent of t, then, due to arbitrariness of $\mathbf{g}(a)$, this could be considered as an independent verification of the conservation of the solenoidal field $\mathbf{u}_0$:

$$\frac{d}{dt}\mathbf{u}_0=\mathbf{0}.$$

The MHD equations expressed in terms of Lagrangian variables become Hamiltonian ones, as in usual mechanics, for momentum $\mathbf{p}=\dot{\mathbf{x}}$ and coordinate $\mathbf{x}$. These variables assign the canonical Poisson structure.

In the Eulerian representation, the MHD equations can be written also in the Hamiltonian form [16],

$$\rho_t=\{\rho,H\},\quad \mathbf{v}_t=\{\mathbf{v},H\},\quad \mathbf{h}_t=\{\mathbf{h},H\},$$

where the noncanonical Poisson bracket $\{F,G\}$ is given by the expression (8). As for ideal hydrodynamics, this Poisson bracket is degenerate. For example, the cross-helicity I_{ch} serves a Casimir for the bracket (8). The reason for the Poisson bracket degeneracy is the same as for one-fluid hydrodynamics—it is connected with a relabeling symmetry of Lagrangian markers.

For the incompressible case, the bracket (8) reduces to one involving only the magnetic field $\mathbf{h}$ and the vorticity $\mathbf{\Omega}$:

$$\{F,G\}=\int\left(\mathbf{\Omega}\left[\operatorname{curl}\frac{\delta F}{\delta\mathbf{\Omega}}\times\operatorname{curl}\frac{\delta G}{\delta\mathbf{\Omega}}\right]\right)d\mathbf{r}+\int\left\{\mathbf{h}\left(\left[\operatorname{curl}\frac{\delta F}{\delta\mathbf{h}}\times\operatorname{curl}\frac{\delta G}{\delta\mathbf{\Omega}}\right]-\left[\operatorname{curl}\frac{\delta G}{\delta\mathbf{h}}\times\operatorname{curl}\frac{\delta F}{\delta\mathbf{\Omega}}\right]\right)\right\}d\mathbf{r}. \quad (65)$$

This Poisson bracket is also degenerate.

IX. VARIATIONAL PRINCIPLE FOR INCOMPRESSIBLE MHD

By analogy with incompressible hydrodynamics, one can introduce the magnetic line representation:

$$\mathbf{h}(\mathbf{r},t)=\int\delta(\mathbf{r}-\mathbf{R}(\mathbf{a},t))[\mathbf{h}_0(\mathbf{a})\cdot\nabla_{\mathbf{a}}]\mathbf{R}(\mathbf{a},t)d\mathbf{a}.$$

If initially $\mathbf{R}(\mathbf{a},0)=\mathbf{a}$, then $\mathbf{h}_0(\mathbf{a})$ is the initial magnetic field. There exist some curvilinear coordinates $\nu_1(\mathbf{a}),\nu_2(\mathbf{a}),s(\mathbf{a})$ such that $\mathbf{r}=\mathbf{R}(\nu_1,\nu_2,s,t)$ determines the position of the magnetic line if ν_1 and ν_2 are fixed. In this case $\nu=(\nu_1,\nu_2)$ can be considered as the marker of this line.

For vorticity, the analog of the vortex line parametrization (28) can be obtained in the regular way as a limit $\epsilon\to 0$ of the corresponding representations for the two-fluid system. Simple calculations give [25]

$$\mathbf{\Omega}(\mathbf{r},t)=\int d\mathbf{a}\,\delta(\mathbf{r}-\mathbf{R}(\mathbf{a},t))[\boldsymbol{\omega}(\mathbf{a},t)\cdot\nabla_a]\mathbf{R}(\mathbf{a},t),$$

where we introduce the notation

$$\boldsymbol{\omega}(\mathbf{a},t)=\mathbf{\Omega}_0(\mathbf{a})+\operatorname{curl}_{\mathbf{a}}[\mathbf{h}_0(\mathbf{a})\times\mathbf{U}(\mathbf{a},t)]. \quad (66)$$

The latter expression is related to the Weber-type transformation (48) but for vorticity: $\mathbf{\Omega}_0(\mathbf{a})$ represents a contribution from the Lagrangian invariant $\mathbf{u}_0(\mathbf{a})$ in Eq. (48),

$$\mathbf{\Omega}_0(\mathbf{a})=\operatorname{curl}\mathbf{u}_0(\mathbf{a}),$$

and, respectively, the last term in Eq. (66) appears from the third term in Eq. (48). The field $\mathbf{U}(\mathbf{a},t)$, however, is not assumed to be solenoidal, while the Jacobian of mapping $\mathbf{r}=\mathbf{R}(\mathbf{a},t)$ is not equal identically to unity. As to conservation of all volumes inside closed magnetic surfaces, it is not a set of constraints but it follows from the equations of motion for $\mathbf{R}$ and $\mathbf{U}$.

From the corresponding limit of the two-fluid system to incompressible MHD, it is possible also to get the expression for the Lagrangian,

$$L=\int\{[(\mathbf{h}_0\cdot\nabla_{\mathbf{a}})\mathbf{R}\times(\mathbf{U}\cdot\nabla_{\mathbf{a}})\mathbf{R}]\mathbf{R}_t\}d\mathbf{a}+1/3\int\{[\mathbf{R}_t\times\mathbf{R}](\mathbf{\Omega}_0\cdot\nabla_{\mathbf{a}})\mathbf{R}\}d\mathbf{a}-\mathcal{H}\{\mathbf{\Omega}\{\mathbf{R},\mathbf{U}\},\mathbf{h}\{\mathbf{R}\}\}. \quad (67)$$

The Hamiltonian of the incompressible MHD $\mathcal{H}_{\text{MHD}}$ in terms of $\mathbf{U}(\mathbf{a},t)$ and $\mathbf{R}(\mathbf{a},t)$ takes the form

$$\mathcal{H}_{\text{MHD}}=\frac{1}{8\pi}\int\frac{\{[\mathbf{h}_0(\mathbf{a})\cdot\nabla_{\mathbf{a}}]\mathbf{R}(\mathbf{a})\}^2}{\det\|\partial\mathbf{R}/\partial\mathbf{a}\|}d\mathbf{a}+\frac{1}{8\pi}\int\int\frac{\{[(\boldsymbol{\omega}(\mathbf{a}_1)\cdot\nabla_1]\mathbf{R}(\mathbf{a}_1)[\boldsymbol{\omega}(\mathbf{a}_2)\cdot\nabla_2]\mathbf{R}(\mathbf{a}_2)\}}{|\mathbf{R}(\mathbf{a}_1)-\mathbf{R}(\mathbf{a}_2)|}\times d\mathbf{a}_1 d\mathbf{a}_2.$$

Equations of motion for $\mathbf{U}$ and $\mathbf{R}$ follow from the variational principle for the action with the Lagrangian (67):

$$[(\mathbf{h}_0\cdot\nabla_{\mathbf{a}})\mathbf{R}\times\mathbf{R}_t]\cdot(\partial\mathbf{R}/\partial a_\lambda)=-\delta\mathcal{H}/\delta U_\lambda,$$

$$\{[\boldsymbol{\omega}(\mathbf{a},t)\cdot\nabla_{\mathbf{a}}]\mathbf{R}\times\mathbf{R}_t\}-[(\mathbf{h}_0\cdot\nabla_{\mathbf{a}})\mathbf{R}\times(\mathbf{U}_t\cdot\nabla_{\mathbf{a}})\mathbf{R}]=\delta\mathcal{H}/\delta\mathbf{R}.$$

These equations can be obtained also directly from the MHD system (5)–(7) by the same scheme as was used for ideal hydrodynamics.

Thus, we have a variational principle for the MHD-type equations for two solenoidal vector fields. Their topological properties are fixed by $\mathbf{\Omega}_0(\mathbf{a})$ and $\mathbf{h}_0(\mathbf{a})$. These quantities represent Casimirs for the initial Poisson bracket (65). It is worth noting that the obtained equations of motion are gauge invariant. The Lagrangian (67) has a remaining symmetry connected with relabeling of Lagrangian markers of magnetic lines in a two-dimensional manifold which can always be specified locally. Coordinates of this manifold enumerate magnetic lines. This symmetry leads to conservation of volume of magnetic tubes including infinitesimally small magnetic tubes, namely, magnetic lines. This property is analogous to conservation of volumes of vortex tubes in the system (37). This explains why the Jacobian of the mapping $\mathbf{r}=\mathbf{R}(\mathbf{a},t)$, which determines magnetic-field dynamics, can differ from unity.

ACKNOWLEDGMENTS

The authors thank A.B. Shabat for useful discussions of the connection between NLSE and Eq. (24), which resulted

in an integrability declaration for Eq. (44). This work was supported by the Russian Foundation of Basic Research under Grant No. 97-01-00093 and by the Russian Program for Leading Scientific Schools (Grant No. 96-15-96093). The work of E.K. was partially supported by the INTAS, Grant No. 96-0413, and the work of V.R. was partially supported by a Landau Scholarship Grant.

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